RESEARCH STATEMENT

1. Introduction

The focus of my research is on investigating questions in representation theory, often from a geometric perspective. More specifically, my work has centered around the representation theory of **reductive algebraic groups** over fields of positive characteristic, and certain related structures such as **quantum groups**, **Frobenius kernels**, **(restricted) Lie algebras** and **hyperalgebras**. My work also deals with associated topics in geometry, such as the study of **perverse sheaves**, **parity sheaves**, **exotic t-structures**, the **nilpotent cone** (and **Springer resolution**), the **affine Grassmannian**, the **affine flag variety**, and the **semi-infinite flag variety**. I am also interested in areas of combinatorics which serve as an interface between the representation theoretic and geometric sides of my research, this includes various combinatorial aspects of **Kazhdan–Lusztig theory**, such as the theory of **Kazhdan–Lusztig cells** and **p-Kazhdan–Lusztig theory**. Particular instances of the connection between representation theory, geometry and combinatorics can be seen in my publications on the **Humphreys conjecture** (see [Ha3], [AHR2], and §2.1 below), and on the **Lusztig–Vogan bijection** (see [AHR1], [AH], and §2.3 below).

An important example of a reductive algebraic group is the general linear group, denoted $GL_n(\Bbbk)$, consisting of all invertible $n \times n$ matrices with entries in a field $\Bbbk$. Its Lie algebra, denoted $\mathfrak{gl}_n(\Bbbk)$, is given by the set of all $n \times n$ matrices. Informally, a representation of an algebraic group is simply a vector space on which the group acts by linear transformations. If the group is reductive and the field $\Bbbk$ has characteristic 0 (e.g. $\Bbbk = \mathbb{C}$), then any finite-dimensional representation of the group, or its Lie algebra, can be completely decomposed into a direct sum of irreducible representations. This implies that every representation is uniquely determined by a combinatorial invariant known as its character – a certain multivariable Laurent polynomial which generalizes the vector-space dimension. An explicit formula for the character of any irreducible representation was obtained by Hermann Weyl in the early twentieth century, it is now known as Weyl's character formula.

The situation when $\Bbbk$ is a field of characteristic $p > 0$ is considerably different. For instance, there exist non-irreducible finite-dimensional representations which are indecomposable. This implies that the Ext^*-cohomology between representations is generally non-zero. Also, the problem of computing the characters of the irreducible representations is significantly more difficult, and still remains open in many cases. In geometric representation theory, we attempt to address some of these problems, such as computing Ext^*-cohomology and the characters of irreducible representations, by translating them into questions concerning sheaves on various algebraic varieties. This is done through the use of categorical equivalences which identify representations with certain types of sheaves, and maps between representations with morphisms between these sheaves. Important examples include the category of equivariant coherent sheaves on the cotangent bundle to the flag variety, as well as categories of constructible sheaves on the affine Grassmannian and the affine flag variety.

Conversely, let $\widetilde{\mathcal{N}}$ denote the cotangent bundle to the flag variety and let $\mathcal{N} \subset \mathfrak{g}$ denote the cone of nilpotent elements in $\mathfrak{g}$. (If $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$, then $\mathcal{N}$ is the subvariety consisting of all nilpotent $n \times n$ matrices.) There is a famous resolution of singularities $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$, called the Springer resolution. The study of equivariant sheaves on these spaces, and their behavior with respect to push-forward along the Springer resolution is of intrinsic interest. In particular, an avenue of research – known as **Springer theory** – focuses on certain categories of *constructible* sheaves on $\mathcal{N}$ which control the representation theory of the associated Weyl group (e.g. the symmetric group if $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$). On the other hand, much of my research instead deals with the study of *coherent* sheaves on these spaces, which can be called **coherent Springer theory**. The motivation for considering coherent sheaves is that they are known to control the representation theory of the corresponding algebraic group (as opposed the Weyl group). In the characteristic 0 setting, the coherent sheaves instead control the representation theory of Lusztig's divided powers quantum group.

To be more precise, the categorical equivalences mentioned above appear in the following diagram

$$D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_{(I^\vee)}(\mathrm{Gr}_{G^\vee}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{[\mathrm{AR}]} D^{\mathrm{b}}\mathsf{Rep}_{\emptyset}(\mathbf{G}) \xleftarrow{[\mathrm{AMRW}]} D^{\mathrm{b}}\mathsf{Perv}^{\mathrm{mix}}_{\mathcal{IW}}(\mathrm{Fl}_{G^\vee}). \tag{1.1}$$

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A few brief remarks:

(1) The group $G^\vee$ is the Langlands dual group to G (e.g. $G^\vee = GL_n(\mathbb{C})$ if $G = GL_n(\Bbbk)$).
(2) The affine Grassmannian $\mathrm{Gr}_{G^\vee}$ admits a stratification by Iwahori orbits which are indexed by the weight lattice $\mathbf{X}$ for G.
(3) The left-most category in this diagram consists of "mixed" Iwahori-constructible perverse sheaves on $\mathrm{Gr}_{G^\vee}$.
(4) The category second from the left is the main focus of coherent Springer theory.
(5) The category second from the right is the "extended principal" block of $\mathbf{G}$, where $\mathbf{G}$ is the "inverse Frobenius twist" of G (i.e. the Frobenius morphism can be given by $\mathrm{Fr} : \mathbf{G} \to G$).
(6) The right-most category is consists of a certain class of "mixed" Iwahori-constructible sheaves on the affine flag variety $\mathrm{Fl}_{G^\vee}$ called *Iwahori–Whittaker sheaves.*
(7) The middle and right-most functors are technically "degrading functors", in other words, they give equivalences to a certain graded version of the principal block.

1.1. **Overview.** The remainder of this statement will be divided into three sections, each corresponding to an ongoing research program. A short summary of each of these is given below.

Tilting module cohomology and coherent Springer theory (§2). This section gives an overview of an ongoing research program into coherent Springer theory. Of particular interest are the sheaves on $\widetilde{\mathcal{N}}$ which correspond to tilting modules. In this setting, the Humphreys conjecture on support varieties of tilting modules (see §2.1), can be reformulated into a purely geometric conjecture which predicts the supports of the push-forwards of these sheaves. This will be explained in §2. In addition, I will formulate a stronger conjecture – inspired by the results in characteristic 0 due to Bezrukavnikov in [B] – on how these push-forwards behave when restricted to certain nilpotent orbits.

Recent advances that I have made towards verifying these conjectures will also be summarized. This includes a verification of the Humphreys conjecture for $SL_n(\Bbbk)$ when $p > n$, and for reductive algebraic groups of any type for "sufficiently large" p (see [AHR2]). Another important development is a result which establishes a link between the "complex" and "modular" Lusztig–Vogan bijection (see [AHR1]), as well as some interesting new calculations involving perverse coherent sheaves on $\mathcal{N}$ (see [AH]).

Thick tensor ideals and p-Kazhdan–Lusztig theory (§3). This section outlines research into understanding the tensor structure of tilting modules. The objects of study in this program are the thick tensor ideals of tilting modules (see [O2], [An]), and the closely related p-Kazhdan–Lusztig cells of the affine Weyl group (cf. [AHR2, §5], [Je]). A conjectured parameterization of these objects in terms of the tensor structure of the nilpotent centralizers will be proposed. It turns out that the characteristic 0 analogue of this conjecture is true, and in fact, gives an instance of the *Lusztig bijection* between two-sided cells and nilpotent orbits (see [Lu1]). Thus, the proposed bijection can be regarded as a "p-cell" analogue of the Lusztig bijection.

Geometry and diagrammatics for G_1T-modules (§4). This section gives an overview of a program which aims to apply geometric and diagrammatic methods to the representation theory of G_1T (equivalently the study of $\mathbf{X}$-graded $\mathfrak{g}$-modules). A construction of a p-canonical basis for Lusztig's *periodic Hecke module* will be given. Additionally, a potential diagrammatic approach, similar to what has already been achieved for G in [RW2], [AMRW], is sketched. If successful, it should be possible to obtain combinatorial formulas for the characters of the irreducible modules of G when $p > h$, analogous to the formulas obtained in [JW] for tilting module characters. A possible application of these ideas to Donkin's lifting conjecture[1] is also noted (see [D, 2.2])).

2. Tilting module cohomology and coherent Springer theory

Let $\mathfrak{g}$ denote the Lie algebra of G (i.e. the Frobenius twist of the Lie algebra of $\mathbf{G}$). The Frobenius kernel $\mathbf{G}_1 \trianglelefteq \mathbf{G}$ is the subgroup scheme given by the kernel of the Frobenius morphism $\mathrm{Fr} : \mathbf{G} \to G$. Fix an action of $\mathbb{G}_\mathrm{m}$ on $\mathfrak{g}$ by $z \cdot x = z^{-2}x$ for any $z \in \mathbb{G}_\mathrm{m}$ and $x \in \mathfrak{g}$. This induces an action on $\mathcal{N}$ which gives $\Bbbk[\mathcal{N}]$ an even grading concentrated in non-negative degrees. By a classical result of Andersen–Jantzen ([AJ]), there

[1]This conjecture is equivalent the classical Humphreys–Verma conjecture.

exists an isomorphism of graded algebras

$$\mathrm{Ext}^*_{\mathbf{G}_1}(\Bbbk, \Bbbk) \cong \Bbbk[\mathcal{N}].$$

(As a consequence, the cohomology ring for $\mathbf{G}_1$ is concentrated in even degrees.)

Thus, for any two finite-dimensional $\mathbf{G}_1$-modules M, N, the cohomology $\mathrm{Ext}^*_{\mathbf{G}_1}(M, N)$ has the structure of a finitely-generated $\Bbbk[\mathcal{N}]$-module. This module is G-equivariant if M, N have the structure of $\mathbf{G}$-modules. For any module M, let $V_{\mathbf{G}_1}(M)$ be the (set-theoretic) support of $\mathrm{Ext}^*_{\mathbf{G}_1}(M, M)$, and let $\overline{V}_{\mathbf{G}_1}(M)$ be the (set-theoretic) support of $\mathrm{Ext}^*_{\mathbf{G}_1}(\Bbbk, M)$. Thus, if M is the restriction of a $\mathbf{G}$-module, then these subvarieties must be given by a union of G-orbits (see [NPV] for an overview on support varieties in this setting).

2.1. **The Humphreys conjecture on support varieties of tilting modules.** For any dominant weight $\lambda \in \mathbf{X}^+$, let $\mathsf{T}(\lambda)$ denote the corresponding indecomposable *tilting module* (see [Ja, Appendix E]). The tilting modules form an important class of representations for $\mathbf{G}$, and are known to be related to parity sheaves on $\mathrm{Gr}_{G^\vee}$ by the work of Juteau–Mautner–Williamson in [JMW]. They are also stable under tensor product, and form an additive monoidal subcategory $\mathsf{Tilt}(G) \subset \mathsf{Rep}(G)$. In the 1990s, J. Humphreys conjectured a description of the $V_{\mathbf{G}_1}(\mathsf{T}(\lambda))$ (cf. [Hu1]). An equivalent formulation of this conjecture involving the $\overline{V}_{\mathbf{G}_1}(\mathsf{T}(w_\lambda \cdot 0))$ for $\lambda \in \mathbf{X}^+$ was later given by Achar, Riche and myself in [AHR2, Lemma 8.11]. To formulate this conjecture, we will first need some preliminaries.

The affine Weyl group W_{aff} admits a decomposition into two-sided Kazhdan–Lusztig cells (see [Hu2]). These cells parameterize a certain class of two-sided ideals in the associated affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$. In [Lu1], a combinatorial bijection, known as the *Lusztig bijection*, between two-sided cells and G-orbits in $\mathcal{N}$ was given. Now by [LX], intersecting two-sided cells of W_{aff} with ${}^0W_{\mathrm{aff}}$ provides a bijection between the set of right cells of W_{aff} which intersect ${}^0W_{\mathrm{aff}}$, and the set of two-sided cells of W_{aff}. These particular intersections are called *anti-spherical right cells*, and parameterize are certain class of submodules of the *anti-spherical module* of $\mathcal{H}_{\mathrm{aff}}$ (see §3 below). Combining with the Lusztig bijection gives a bijection between anti-spherical right cells and nilpotent orbits.

The anti-spherical right cells, in particular, give a partition of $\mathbf{X}$ since there is a natural isomorphism $\mathbf{X} \cong {}^0W_{\mathrm{aff}}$ which identifies $\lambda \overset{\sim}{\leftrightarrow} w_\lambda$. Restricting this isomorphism to the set of dominant weights $\mathbf{X}^+ \subset \mathbf{X}$ compatibly gives $\mathbf{X}^+ \cong {}^0W^0_{\mathrm{aff}}$. For each orbit $C \subset \mathcal{N}$, let $\mathbf{X}_C \subset \mathbf{X}$ denote the corresponding anti-spherical right cell given by the Lusztig bijection. It is known that $\mathbf{X}_C^+ := \mathbf{X}_C \cap \mathbf{X}^+ \neq \emptyset$ for any $C \subset \mathcal{N}$.

Conjecture 2.1 (Humphreys conjecture)**.** *Let $C \subset \mathcal{N}$ and $\lambda \in \mathbf{X}_C^+$, then*

$$\overline{V}_{\mathbf{G}_1}(\mathsf{T}(w_\lambda \cdot 0)) = \overline{C}.$$

Informally, the Lusztig bijection can be realized by taking support varieties of tilting modules.

The preceding conjecture has yet to be completely verified. However, in the last few years I have managed to make some fairly substantial progress. For instance, the results in my thesis ([Ha3]), as well as [AHR2], can be combined to give the following theorem.

Theorem 2.2 (Hardesty)**.** *Conjecture 2.1 holds for simple algebraic group G of type A with $p > h$ (e.g. $G = SL_n$ and $p > n$).*

So at the moment, the conjecture remains open only for groups not of type A. But in [Ha3], I, along with Achar and Riche, still made significant progress towards the conjecture for groups of arbitrary type by utilizing a number of recent developments in geometric representation theory (see §2.2 below for the geometric approach). In particular, we obtained the following theorem.

Theorem 2.3 (Achar–Hardesty–Riche)**.** *Let $\mathbf{G}$ be an arbitrary reductive algebraic group, then there exists some integer $M > 0$ such that Conjecture 2.1 holds for all $p > M$.*

Remark 2.4. Unfortunately, there is no explicit bound on the integer M appearing above. In addition, we were at least able to prove that $\overline{V}_{\mathbf{G}_1}(\mathsf{T}(w_\lambda \cdot 0)) \supseteq \overline{C}$ whenever $p > h$, it is the argument for other inclusion "$\subseteq$", which requires the strong lower bound on p.

2.2. **Exotic parity objects.** The category $D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ is known to possess an *exotic t-structure* (see [MR]), whose heart $\mathsf{ExCoh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$, is *graded highest weight* (in the sense of [CPS2]), with highest weight poset $(\mathbf{X}, \leq)$ where $\leq$ denotes the "convex order" (see [AR, §9.4]). Furthermore, there is an equivalence $D^{\mathrm{b}}\mathsf{ExCoh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \cong D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$.

In [AHR2, §2.2], a class of *parity objects* for $D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ were studied. These objects can be characterized as the "abstract" parity objects of $D^{\mathrm{b}}\mathsf{ExCoh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ in the sense of [CPS1]. Alternatively, they are the objects corresponding to the tilting sheaves of $\mathsf{Perv}^{\mathrm{mix}}_{(I^\vee)}(\mathrm{Gr}_{G^\vee}, \Bbbk)$ under the left-most equivalence in (1.1). The indecomposable even parity objects are indexed by $\mathbf{X}$, and will be denoted by $\widetilde{\mathcal{E}}(\lambda)$. The irreducible objects will be denoted by $\widetilde{\mathcal{V}}(\lambda)$.

If $p > h$, then by [AR, Theorem 1.5], the $\widetilde{\mathcal{E}}(\lambda)$ are sent to $\mathsf{T}(w_\lambda \cdot 0)$ under the middle functor of (1.1). Additionally, the cohomology $\mathrm{Ext}^*_{\mathbf{G}_1}(\Bbbk, \mathsf{T}(w_\lambda \cdot 0))$ naturally identifies with the cohomology of the sheaf $\pi_*\widetilde{\mathcal{E}}(w_0\lambda)$, where $\pi_* : D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ is induced from the Springer resolution. In particular, $\overline{V}_{\mathbf{G}_1}(\mathsf{T}(w_\lambda \cdot 0)) = \operatorname{supp} \pi_*\widetilde{\mathcal{E}}(w_0\lambda)$. It now follows that the conjecture below is equivalent to Conjecture 2.1 when $p > h$.

Conjecture 2.5. *Let $C \subset \mathcal{N}$ and $\lambda \in \mathbf{X}^+_C$ be arbitrary, then* $\operatorname{supp} \pi_*\widetilde{\mathcal{E}}(w_0\lambda) = \overline{C}$.

Suppose now that the preceding conjecture holds, then for any $\lambda \in \mathbf{X}^+_C$, the orbit C is open in the support of $\pi_*\widetilde{\mathcal{E}}(w_0\lambda)$ and (informally) the restriction $\pi_*\widetilde{\mathcal{E}}(w_0\lambda)|_C$ gives a sheaf on the orbit C. This restriction identifies with an object in the (derived) module category for the centralizer $G^x \subseteq G$ for any $x \in C$.

It is well-known that G^x admits a Levi decomposition $G^x = G^x_{\mathrm{red}} \ltimes G^x_{\mathrm{unip}}$, where G^x_{red} is a (possibly disconnected) reductive group. In [AHR3], methods from classical Clifford theory were used to construct a highest weight structure on $\mathsf{Rep}(G^x_{\mathrm{red}})$. So this category possesses tilting modules which, as in the connected case, are stable under tensor product. Under the identification of $\mathsf{Rep}(G^x)$ with $\mathsf{Coh}^G(C)$, the corresponding modules can be referred to as sheaves. In particular, by [Ac], for any $\lambda \in \mathbf{X}^+$ $\pi_*\widetilde{\mathcal{V}}(w_0\lambda)|_C$ is an irreducible sheaf whenever $\operatorname{supp} \pi_*\widetilde{\mathcal{V}}(w_0\lambda) = \overline{C}$, and every irreducible sheaf on C is obtained in this way.

Conjecture 2.6. *For $\lambda \in \mathbf{X}^+_C$, $\pi_*\widetilde{\mathcal{E}}(w_0\lambda)$ is an indecomposable object of $D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, and the restriction $\pi_*\widetilde{\mathcal{E}}(w_0\lambda)|_C$ is an indecomposable tilting sheaf on C which is indexed by the same "weight" as the irreducible sheaf $\pi_*\widetilde{\mathcal{V}}(w_0\lambda)|_C$.*

Some remarks:

(1) If $p > h$, then it is not difficult to verify this conjecture for all $\lambda \in \mathbf{X}^+_{\{0\}}$ by explicitly computing $\mathrm{Ext}^*_{\mathbf{G}_1}(\Bbbk, \mathsf{T}(w_\lambda \cdot 0))$ and applying the techniques from [D].
(2) This conjecture can also be verified for the principal nilpotent orbit.
(3) The results of [B] in characteristic 0, where it was proven that $\widetilde{\mathcal{E}}(\lambda) = \widetilde{\mathcal{V}}(\lambda)$ for all $\lambda \in \mathbf{X}$, are consistent with this conjecture.

2.3. **The modular Lusztig–Vogan bijection.** The category $D^{\mathrm{b}}\mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ has a *perverse coherent t-structure* whose heart is denoted $\mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ (see [Ac]). The functor π_* is t-exact with respect to the exotic and perverse t-structures, and thus induces a functor $\pi_* : \mathsf{ExCoh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \to \mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$. The irreducible objects of $\mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ are all of the form $\mathcal{V}(\lambda) := \pi_*\widetilde{\mathcal{V}}(w_0\lambda)$ for $\lambda \in \mathbf{X}^+$, where $\pi_*\widetilde{\mathcal{V}}(w_0\lambda) = 0$ for $\lambda \notin \mathbf{X}^+$. On the other hand, the machinery of perverse coherent sheaves produces for each orbit C, a functor

$$\mathcal{IC}(C, -) : \mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(C) \to \mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N}).$$

For any irreducible sheaf $\mathsf{L} \in \mathsf{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(C)$, the object $\mathcal{IC}(C, \mathsf{L}) \in \mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ is irreducible, and furthermore, every irreducible object is of this form. The cohomology of these was explicitly computed in the case of PGL_3 by Achar and myself in [AH].

Now the two equivalent descriptions of the irreducible objects of $\mathsf{PCoh}^{\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ leads to the following bijection, called the $\Bbbk$-*Lusztig–Vogan bijection*,

$$\mathbf{X}^+ \xleftrightarrow{\sim} \{(C, \mathsf{L}) \mid C \subset \mathcal{N},\ \mathsf{L} \in \mathsf{Coh}^G(C) \text{ an irreducible sheaf}\}.[2] \tag{2.1}$$

[2]In the characteristic 0 case, this is simply called the *Lusztig–Vogan bijection*.

Remark 2.7. In [AH, Theorem 4.5], we also worked out a description of this bijection in the $G \times \mathbb{G}_m$-equivariant setting, which confirmed a conjecture of Ostrik given in [O1].

It is not at all clear from the definition whether this bijection is independent of the characteristic of the field. In fact, the problem of relating the characteristic 0 bijection to the positive characteristic bijection is intrinsically interesting, as well as a significant step towards proving Conjecture 2.5 and Conjecture 2.6.

I recently obtained a proof of this independence in joint work with P. Achar and S. Riche (see [AHR1]). The basic strategy was to extend the theory of exotic t-structures to the case where $\widetilde{\mathcal{N}}$ is defined over a complete discrete valuation ring $\mathbb{O}$, with fraction field $\mathbb{K}$ and characteristic p residue field $\mathbb{F}$ which we assume to be algebraically closed. In this setup, there is a subcategory of "torsion-free" objects on which the base-change functors to $\widetilde{\mathcal{N}}_{\mathbb{K}}$ and $\widetilde{\mathcal{N}}_{\mathbb{F}}$ are t-exact. This allows us to relate the exotic t-structures for $\mathbb{K}$ and $\mathbb{F}$.

The exotic t-structure machinery can then be employed to construct certain "lattices" inside the irreducible exotic sheaves on $\widetilde{\mathcal{N}}_{\mathbb{K}}$, denoted $\widetilde{\mathcal{V}}_{\mathbb{K}}(\lambda)$ for $\lambda \in \mathbf{X}$. In particular, there is a notion of "minimal" and "maximal" admissible lattices whose base changes to $\widetilde{\mathcal{N}}_{\mathbb{F}}$ are objects of $\mathsf{ExCoh}^{\mathbb{G}_m}(\widetilde{\mathcal{N}})$, denoted $\widetilde{\Delta}^{\mathrm{red}}(\lambda)$ and $\widetilde{\nabla}^{\mathrm{red}}(\lambda)$ respectively. To compare the $\mathbb{K}$ and $\mathbb{F}$ Lustig–Vogan bijections, it suffices to compare the $\mathcal{V}_{\mathbb{F}}(\lambda) := \pi_* \widetilde{\mathcal{V}}(w_0\lambda)$ to $\Delta^{\mathrm{red}}_{\mathbb{F}}(\lambda) := \pi_* \widetilde{\Delta}^{\mathrm{red}}(w_0\lambda)$ and $\nabla^{\mathrm{red}}_{\mathbb{F}}(\lambda) := \pi_* \widetilde{\nabla}^{\mathrm{red}}(w_0\lambda)$.

Such a comparison requires a theory of *balanced nilpotent sections* $x \in \mathfrak{g}_{\mathbb{O}}$, which can be thought of as "nice" $\mathbb{O}$-points inside the $\mathbb{K}$ nilpotent orbits (see [Ha2]). As well as some deep structural results on the centralizer $G^x \subseteq G$ of such a section (G is taken to be a group scheme over $\mathbb{O}$ in this case). The necessary results were all obtained by myself in [Ha2], and are summarized in the following theorem.

Theorem 2.8 (Hardesty)**.** *Let $x \in \mathfrak{g}_{\mathbb{O}}$ be a balanced nilpotent section.*

(1) The morphism $G^x \to \mathrm{Spec}(\mathbb{O})$ is smooth.
(2) The (discrete) component groups $G^x_{\overline{\mathbb{K}}}$ and $G^x_{\mathbb{F}}$ are isomorphic,
(3) There exists a normal subgroup scheme $(G^x)^\circ \trianglelefteq G^x$ such that the quotient group scheme $A(x) := G^x/(G^x)^\circ$ is constant, where $A(x)(\overline{\mathbb{K}})$ and $A(x)(\mathbb{F})$ are the component groups of the geometric fibers.

The importance of the preceding theorem lies in the fact that when combined with the constructions from [AHR3], it becomes possible to simultaneously parametrize the irreducible representations for the $\mathbb{K}$ and $\mathbb{F}$ centralizers in a way which is consistent with their Clifford theoretic parameterizations. The main results of [AHR1] are now summarized by the following theorem.

Theorem 2.9 (Achar–Hardesty–Riche)**.** *Let $C_{\mathbb{K}} \subset \mathcal{N}_{\mathbb{K}}$ and $C_{\mathbb{F}} \subset \mathcal{N}_{\mathbb{F}}$ be a pair of orbits with the same Bala–Carter label, and let $x \in \mathcal{N}_{\mathbb{O}}(\mathbb{O})$ be the balanced nilpotent section. For any $\lambda \in \mathbf{X}^+_C$,*

(1) $\mathrm{supp}\, \Delta^{\mathrm{red}}(\lambda) = \mathrm{supp}\, \nabla^{\mathrm{red}}(\lambda) = \overline{C_{\mathbb{F}}}$*,*
(2) $\mathrm{supp}\, \mathcal{V}_{\mathbb{F}}(\lambda) = \overline{C_{\mathbb{F}}}$*,*
(3) $\Delta^{\mathrm{red}}(\lambda)|_{C_{\mathbb{F}}}$ and $\nabla^{\mathrm{red}}(\lambda)|_{C_{\mathbb{F}}}$ give the Weyl sheaf and coWeyl sheaf corresponding to the irreducible sheaf $\mathcal{V}_{\mathbb{F}}(\lambda)|_{C_{\mathbb{F}}}$.

In particular, the $\mathbb{K}$ and $\mathbb{F}$ Lusztig–Vogan bijections coincide by the third statement.

The following corollary on the highest weight structure is also worth noting.

Corollary 2.10 (Achar–Hardesty–Riche)**.** *The restriction of $\leq$ from $\mathbf{X}^+$ to $\mathbf{X}^+_C$ also gives a highest weight order on* $\mathsf{Rep}(G^{x_{\mathbb{F}}}_{\mathrm{red}})$.

3. Thick tensor ideals and p-Kazhdan–Lusztig theory

As in §2.1, we let $\mathcal{H}_{\mathrm{aff}}$ denote the affine Hecke algebra associated to W_{aff}. Also, let $\mathcal{H}_{\mathrm{f}} \subset \mathcal{H}_{\mathrm{aff}}$ the Hecke algebra associated to the finite Weyl group $W \subset W_{\mathrm{aff}}$. The algebra $\mathcal{H}_{\mathrm{aff}}$ can be described as a free $\mathbb{Z}[t^{\pm 1}]$-module with a basis consisting of H_w for $w \in W_{\mathrm{aff}}$, and multiplicative structure given by [S, §2]. There also exists an important right $\mathcal{H}_{\mathrm{aff}}$-module $\mathcal{M}_{\mathrm{asph}} = (\mathrm{sign}) \otimes_{\mathcal{H}_{\mathrm{f}}} \mathcal{H}_{\mathrm{aff}}$, called the anti-spherical module.[3]

The algebra $\mathcal{H}_{\mathrm{aff}}$ possesses a canonical basis whose elements are denoted by $\underline{H}_w$ for $w \in W_{\mathrm{aff}}$. This also induces a canonical basis of $\mathcal{M}_{\mathrm{asph}}$ given by $\underline{M}_w := 1 \otimes \underline{H}_w$ for $w \in {}^0W_{\mathrm{aff}}$. Now an equivalence relation

[3] This module is denoted by $\mathcal{N}$ in [S, §3]).

$\sim^0$ can be defined on ${}^0W_{\text{aff}}$, where $w \sim^0 y$ if and only if $\underline{M}_w$ and $\underline{M}_y$ generate the same submodule of $\mathcal{M}_{\text{asph}}$. The equivalence classes for this relation are the anti-spherical right cells introduced in §2.1. One can similarly define left, right and two sided cells in W_{aff} by considering the left, right and two-sided ideals of $\mathcal{H}_{\text{aff}}$ generated by $\underline{H}_w$ for $w \in W$.

For each p, there also exists a *p-canonical basis* of $\mathcal{H}_{\text{aff}}$ given by ${}^p\underline{H}_w$ for $w \in W_{\text{aff}}$, and for $\mathcal{M}_{\text{asph}}$ given by ${}^p\underline{M}_w$ for $w \in {}^0W_{\text{aff}}$ (see [JW]). In this setup, the relation $\sim^p$ can analogously be defined on ${}^0W_{\text{aff}}$, where the equivalence classes are called *anti-spherical right p-cells*. Similarly, p-cells also exist for W_{aff} (see [Je] for an overview of p-cell theory).

A *thick tensor ideal* of $\mathsf{Tilt}(\mathbf{G})$ is defined to be a full monoidal subcategory $\mathcal{I} \subseteq \mathsf{Tilt}(\mathbf{G})$, which is closed under direct sums and direct summands, and satisfies $\mathsf{T}(\lambda) \otimes \mathsf{T} \in \mathcal{I}$ for any $\lambda \in \mathbf{X}^+$ and $\mathsf{T} \in \mathcal{I}$. In [AHR2, §7.3], an equivalence relation $\sim^{\mathsf{T}}$ was defined on $\mathbf{X}^+$, where $\lambda \sim^{\mathsf{T}} \mu$ if and only if $\mathsf{T}(\lambda)$ and $\mathsf{T}(\mu)$ generate the same thick tensor ideal. If ${}^0W_{\text{aff}}$ is identified with the set of alcoves of $\mathbf{X}^+$ (under the p-dilated "dot action" of W_{aff}), then for each anti-spherical right p-cell, the associated *p-canonical weight cell* of $\mathbf{X}^+$ is given by the union of the lower closures of all alcoves corresponding to the elements of the cell. The following theorem was proven by Achar, Riche and myself in [AHR2, §7.3].

Theorem 3.1 (Achar–Hardesty–Riche)**.** *The equivalence classes for $\sim^{\mathsf{T}}$ coincide with the p-canonical weight cells.*

Remark 3.2. This is a generalization to positive characteristic of a result by Ostrik in [O2].

Thus, classifying indecomposable thick tensor ideals of tilting modules is equivalent to classifying right anti-spherical p-cells.

3.1. **Parametrizing p-cells and thick tensor ideals.** The equivalence relations $\sim^0$ and $\sim^p$ are also defined on $\mathbf{X}$ via ${}^0W_{\text{aff}} \cong \mathbf{X}$. For this rest of this section these equivalence classes will be simply called the 0-*cells* and *p-cells* of $\mathbf{X}$. Let $\mathbf{c}^0(\lambda) \subset \mathbf{X}$ and $\mathbf{c}^p(\lambda) \subset \mathbf{X}$ denote the unique 0-cell and unique p-cell containing $\lambda \in \mathbf{X}$ respectively (in some cases, it will be more convenient to simply write $\mathbf{c}^p$ and $\mathbf{c}^0$).

As in [Ja, II.H.7], let $\mathcal{U}_\zeta(\mathfrak{g}_{\mathbb{C}})$ denote the Lusztig quantum group group for $\mathfrak{g}_{\mathbb{C}}$ at a primitive p^{th} root of unity $\zeta \in \mathbb{C}$. In [O2], a bijection between 0-cells and indecomposable thick tensor ideals of tilting modules for $\mathcal{U}_\zeta(\mathfrak{g}_{\mathbb{C}})$ was established. Combining this bijection with the Lusztig bijection gives the diagram

$$\begin{tikzcd}
\{\text{0-cells}\} \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\sim"'] & & \{\mathcal{I} \subseteq \mathsf{Tilt}(\mathcal{U}_\zeta(\mathfrak{g}_{\mathbb{C}}))\} \\
& \{\text{nilpotent orbits}\}, \arrow[ur, leftrightarrow, "\sim"'] &
\end{tikzcd} \tag{3.1}$$

where $\mathcal{I} \subseteq \mathsf{Tilt}(\mathcal{U}_\zeta(\mathfrak{g}_{\mathbb{C}}))$ denotes an arbitrary indecomposable thick tensor ideal.

The truth of the Humphreys conjecture for quantum groups, proven in [B], implies that the "diagonal" bijections can be explicitly realized by computing the "relative support varieties" and "classical support varieties" of tilting modules respectively (see [AHR2, §1.1]).

In the p-cell setting, an analogue to the "horizontal" bijection in (3.1) was obtained in [AHR2, §7]. Now the major open problem at the moment is to find a natural replacement for the set of nilpotent orbits. A conjecture for this replacement will be formulated below, but first the following lemma will be required.

Lemma 3.3. *If Conjecture 2.1 holds (i.e. for $\mathbf{G} = SL_n$ and $p > n$, or for $p \gg 0$ in general), then every 0-cell can be decomposed into a disjoint union of p-cells. Precisely, for any orbit $C \subset \mathcal{N}$,*

$$\mathbf{X}_C = \bigsqcup_{\mathbf{c}^p \subseteq \mathbf{X}_C} \mathbf{c}^p. \tag{3.2}$$

This reduces the problem of parametrizing p-cells to determining the decomposition in (3.2). Let $G^C := G^x \subseteq G$ denote the centralizer of some $x \in C$, and let G^C_{red} be the reductive quotient. Recall from Theorem 2.9, and Corollary 2.10, that $\mathsf{Rep}(G^C_{\text{red}})$ is a monoidal highest weight category with weight poset $(\mathbf{X}^+_C, \preceq)$. Now as in [AHR2, §7.3], the thick tensor ideals of $\mathsf{Tilt}(G^C_{\text{red}})$ give a theory of weight cells for $\mathbf{X}^+_C$. Specifically, for any $\lambda \in \mathbf{X}^+_C$, let $\mathsf{T}^C_\lambda \in \mathsf{Tilt}(G^C_{\text{red}})$ denote the corresponding indecomposable tilting module, then there is an equivalence relation $\sim^C$ on $\mathbf{X}^+_C$, where $\lambda \sim^C \mu$ if and only if T^C_λ and T^C_μ generate the same thick tensor ideal. The weight cells are defined to be the equivalence classes for this relation, where for $\lambda \in \mathbf{X}_C^+$, $\mathbf{c}^C(\lambda)$ denotes the unique weight cell containing λ (again, it is often convenient to simply write $\mathbf{c}^C$).

Before formulating the conjecture note that by [AHR2, Lemma 8.8], every p-cell of $\mathbf{X}$ has a nonempty intersection with $\mathbf{X}^+$. This implies that each $\mathbf{c}^p \subseteq \mathbf{X}_C$ appearing in (3.2) is of the form $\mathbf{c}^p(\mu)$.

Conjecture 3.4. *For any $\mu \in \mathbf{X}_C$, $\mathbf{c}^p(\mu) \cap \mathbf{X}_C^+ = \mathbf{c}^C(\mu)$. Thus by Lemma 3.3, for any $\lambda \in \mathbf{X}$ with $\mathbf{c}^0(\lambda) = \mathbf{X}_C$, there is a natural bijection*

$$\{p\text{-cells contained in } \mathbf{c}^0(\lambda)\} \xleftrightarrow{\sim} \{\text{weight cells of } \mathbf{X}_C^+\}. \tag{3.3}$$

Remark 3.5. This conjecture was verified in the special case where $p \geq 2h - 2$ and $C = \{0\}$ in [An], and for the cells corresponding to the principal and subregular orbits in [Ra]. It can also be confirmed in its entirety in the type B_2 case from the calculations in [Je].

The preceding conjecture is summarized by the following diagram

$$\begin{tikzcd}[column sep=small]
\{p\text{-cells}\} \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\sim"'] & & \{\mathcal{I} \subseteq \mathsf{Tilt}(\mathbf{G})\} \arrow[dl, leftrightarrow, "\sim"] \\
& \{\mathcal{I}^C \subseteq \mathsf{Tilt}(G^C_{\mathrm{red}}) \text{ for } C \subset \mathcal{N}\}, &
\end{tikzcd} \tag{3.4}$$

where $\mathcal{I}$ and $\mathcal{I}^C$ denote arbitrary indecomposable thick tensor ideals.

Remark 3.6. This expected picture is consistent with (3.1), since the category $\mathsf{Rep}(G^C_{\mathrm{red}})$ is semisimple for each orbit $C \subset \mathcal{N}$ in characteristic 0.

4. Geometry and diagrammatics for G_1T-modules

Let $\mathbf{G}_1\mathbf{T} \subseteq \mathbf{G}$ be the inverse image of T under Fr (for any subgroup $\mathbf{H} \subseteq \mathbf{G}$, $\mathbf{H}_1\mathbf{T}$ is similarly defined). If $\Bbbk_\lambda$ denotes the corresponding 1-dimensional $\mathbf{B}_1^+\mathbf{T}$-module for $\lambda \in \mathbf{X}$, then the *baby Verma module* is defined by $\mathsf{Z}(\lambda) := \mathrm{CoInd}_{\mathbf{B}_1^+\mathbf{T}}^{\mathbf{G}_1\mathbf{T}} \Bbbk_\lambda$, and the *co-baby Verma module* is defined by $\mathsf{Z}'(\lambda) := \mathrm{Ind}_{\mathbf{B}_1\mathbf{T}}^{\mathbf{G}_1\mathbf{T}} \Bbbk_\lambda$. Also, let $\widehat{\mathsf{L}}(\lambda)$ and $\mathsf{Q}(\lambda)$ be the corresponding irreducible and indecomposable injective module respectively.

The baby Verma, co-baby Verma, and indecomposable injective modules for $\mathbf{G}_1\mathbf{T}$ share a number of key properties in common with the Weyl, coWeyl, and indecomposable tilting modules for $\mathbf{G}$ respectively (see [Ja, Lemma II.9.9, Proposition II.11.2]). In particular, $\mathsf{Rep}(\mathbf{G}_1\mathbf{T})$ behaves very much like a (monoidal) highest weight category with highest weight poset $(\mathbf{X}, \preceq)$. The only issue is that the poset does not admit a refinement to $\mathbb{Z}_{\geq 0}$.

The category $\mathsf{Rep}(\mathbf{G}_1\mathbf{T})$ admits a block decomposition which is compatible with the block decomposition of $\mathsf{Rep}(\mathbf{G})$ (cf. [Ja, II.7.2 and II.9.22]). In particular, the *(extended) principal block* for $\mathbf{G}_1\mathbf{T}$, denoted $\mathsf{Rep}_\emptyset(\mathbf{G}_1\mathbf{T})$, is defined as the Serre subcategory generated by $\widehat{\mathsf{L}}(w \cdot 0)$ for any $w \in W_{\mathrm{aff}}$. Identifying W_{aff} with the orbit $W_{\mathrm{aff}} \cdot 0 \subset \mathbf{X}$, denote

$$\widehat{\mathsf{L}}_w := \widehat{\mathsf{L}}(w \cdot 0), \quad \mathsf{Z}_w := \mathsf{Z}(w \cdot 0), \quad \mathsf{Z}'_w := \mathsf{Z}'(w \cdot 0), \quad \mathsf{Q}_w := \mathsf{Q}(w \cdot 0)$$

for $w \in W_{\mathrm{aff}}$. Under this identification, the lattice order $\preceq$ on $\mathbf{X}$ induces an order on W_{aff}, also denoted $\preceq$. For any $w \in W$, all the composition factors $\widehat{\mathsf{L}}_x$ of Z_w must all satisfy $x \preceq w$.

Determining the composition multiplicities $[\mathsf{Z}_y : \widehat{\mathsf{L}}_x]$ for $x \preceq y$ is a major problem in modular representation theory. In fact, since the characters of the Z_y are known (cf. [Ja, Lemma II.9.2]), this problem is equivalent to determining the characters of the irreducible $\mathbf{G}_1\mathbf{T}$-modules (and hence the irreducible $\mathbf{G}$-modules by [Ja, II.3.17]). Furthermore, computing these composition multiplicities is equivalent to computing the baby Verma filtration multiplicities of the indecomposable injectives due to the reciprocity result $[\mathsf{Z}_y : \widehat{\mathsf{L}}_x] = [\mathsf{Q}_x : \mathsf{Z}_y]$ (see [Ja, Proposition II.11.4]). (Computing the $[\mathsf{Q}_x : \mathsf{Z}_y]$ is highly analogous to computing the good filtration multiplicities of tilting modules.)

For any $x, y \in W_{\mathrm{aff}}$, let $p_{x,y} \in \mathbb{Z}[t^{\pm 1}]$ denote the *periodic Kazhdan–Lusztig polynomial* (see (4.1) below). The following famous theorem is due to Andersen–Jantzen–Soergel, and originally appeared in [AJS].

Theorem 4.1. *Assume $p \gg 0$,[4] then $[\mathsf{Z}_{w_0 x} : \widehat{\mathsf{L}}_{w_0 y}] = p_{x,y}(1)$ for any $x \preceq y$. Furthermore, the degree k coefficient of $p_{x,y}$ actually gives the the multiplicity of $\widehat{\mathsf{L}}_{w_0 x}$ in the k^{th}* radical layer *of $\mathsf{Z}_{w_0 y}$ (see* [Ja, D.13]*).*

Remark 4.2. The singular blocks for $\mathbf{G}_1\mathbf{T}$ are also of interest (see [Ri], [AK], and [NZ]). I explicitly calculated the radical layers of the baby Verma modules in certain singular blocks for $G = GL_n$ in [Ha1].

4.1. **The p-canonical basis of the periodic Hecke module.** In order to obtain an analogue to Theorem 4.1 in smaller characteristics, it will first be necessary to obtain a "p-analogue" of the periodic polynomials which is compatible with the above multiplicity problem. To define these polynomials, we recall another important module of $\mathcal{H}_{\mathrm{aff}}$, called the *periodic Hecke module*, which was introduced by Lusztig in [Lu2]. Following [S, §4], we have

$$\mathcal{P} := \bigoplus_{w \in W_{\mathrm{aff}}} \mathbb{Z}[t^{\pm 1}] P_w,$$

where, as in [Ja, p. 454], the basis element "P_w" is identified with the basis element "A" from via $A = w \cdot A_0$ with A_0 denoting the bottom alcove.

Let $\mathcal{P}^\circ \subset \mathcal{P}$ denote the $\mathcal{H}_{\mathrm{aff}}$ submodule defined in [S, p. 93], so that the *periodic canonical basis* is the basis of $\mathcal{P}^\circ$ appearing in [S, Theorem 4.3(2) and Remark 4.4]. In the present notation, this basis is given by

$$\underline{P}_w = \sum_{w \succeq x} p_{x,w} P_x, \tag{4.1}$$

for $w \in W_{\mathrm{aff}}$, where $p_{w,w} = 1$ and $p_{x,w} \in t\mathbb{Z}[t]$ for all $w \neq x$.

Techniques from coherent Springer theory can now be employed to obtain a p-analogue to the periodic canonical basis. In particular, it is shown in [CG] that $K^{T \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ naturally identifies with $\mathcal{P}$ as an $\mathcal{H}_{\mathrm{aff}}$-module, and therefore, $D^{\mathrm{b}}\mathsf{Coh}^{T \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ can be regarded as a categorification of $\mathcal{P}$.

To see why this is the case, first consider the following commutative diagram:

$$\begin{tikzcd} D^{\mathrm{b}}\mathsf{Coh}^{G \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \arrow[r, "\mathrm{Ind}"] \arrow[d, "\mathrm{For}"] & D^{\mathrm{b}}\mathsf{Rep}_{\emptyset}(\mathbf{G}) \arrow[d, "\mathrm{For}"] \\ D^{\mathrm{b}}\mathsf{Coh}^{T \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \arrow[r, "\mathrm{Loc}"] & D^{\mathrm{b}}\mathsf{Rep}_{\emptyset}(\mathbf{G}_1\mathbf{T}), \end{tikzcd} \tag{4.2}$$

where the functor "Loc" was constructed in [Ri], and the commutativity has been verified by N. Cooney in a currently unpublished manuscript.

Let $\widetilde{\mathsf{Rep}_{\emptyset}}(\mathbf{G}_1\mathbf{T}) \subset D^{\mathrm{b}}\mathsf{Coh}^{T \times \mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ be the abelian subcategory which gets sent to $\mathsf{Rep}_{\emptyset}(\mathbf{G}_1\mathbf{T})$ by "Loc". For each $w \in W_{\mathrm{aff}}$, fix a graded lift ${}^{\mathrm{gr}}\widehat{\mathsf{L}}_w$ of $\widehat{\mathsf{L}}_w$. Also, note that it is possible to obtain graded lifts of the Z_w (and hence Z'_w) by employing the action of the *affine braid group* $\mathbb{B}_{\mathrm{aff}}$ (studied in [BR]). More specifically, set ${}^{\mathrm{gr}}\mathsf{Z}_1 := i_* \mathcal{O}_{\{\overline{1}\} \times \mathfrak{n}^*}$ (i.e. the structure sheaf of the fiber over $\overline{1} \in G/B$), and for any $w \in W_{\mathrm{aff}}$, let

$${}^{\mathrm{gr}}\mathsf{Z}_w := b_w \cdot {}^{\mathrm{gr}}\mathsf{Z}_1,$$

where $b_w \in \mathbb{B}_{\mathrm{aff}}$ is the corresponding braid group element. It can be verified that ${}^{\mathrm{gr}}\mathsf{Z}_w \in \widetilde{\mathsf{Rep}_{\emptyset}}(\mathbf{G}_1\mathbf{T})$. The fact that these sheaves "degrade" to the baby Verma modules roughly follows from the general properties of the $\mathbb{B}_{\mathrm{aff}}$ action, the *linear Koszul duality* between the derived fiber product $\widetilde{\mathfrak{g}} \times^L \{0\}$ and $\widetilde{\mathcal{N}}$ (regarded as a DG-scheme), and the geometric realization of the baby Verma module $\mathsf{Z}_1 = \mathsf{Z}(0)$ as a certain skyscraper sheaf on $\widetilde{\mathfrak{g}} \times^L \{0\}$ (see [BR] and [Ri]). Similarly, the co-baby Vermas ${}^{\mathrm{gr}}\mathsf{Z}'_w$ lift as well, since ${}^{\mathrm{gr}}\mathsf{Z}_w$ and ${}^{\mathrm{gr}}\mathsf{Z}'_w$ should be related through Serre–Grothendieck duality.

The construction of the ${}^{\mathrm{gr}}\mathsf{Z}_w$ via the braid group action can actually be extended to the case where $\widetilde{\mathcal{N}}$ is defined over a discrete valuation ring. Then a standard base change argument produces an $\mathcal{H}_{\mathrm{aff}}$-module isomorphism between the equivariant K-theory in characteristic 0 and the equivariant K-theory in positive characteristic. This reduces the following conjecture to the characteristic 0 case which should be staightforward to verify.

[4]This theorem generally fails for p not *extremely large* relative to h by [W].

Conjecture 4.3. *There is an isomorphism of $\mathcal{H}_{\mathrm{aff}}$-modules between $K^{T\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ and $\mathcal{P}$ given by*

$$K^{T\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} \mathcal{P}, \quad [{}^{\mathrm{gr}}\mathsf{Z}_{w_0 w}] \mapsto P_w$$

for all $w \in W_{\mathrm{aff}}$.

Definition 4.4. For any $w \in W_{\mathrm{aff}}$, let ${}^p\underline{P}_w \in \mathcal{P}$ correspond to the class $[{}^{\mathrm{gr}}\widehat{\mathsf{L}}_{w_0 w}]$ under the isomorphism from Conjecture 4.3, and write

$${}^p\underline{P}_w = \sum_{w \succeq x} {}^p p_{x,w} P_w,$$

where ${}^p p_{x,w} = [{}^{\mathrm{gr}}\mathsf{Z}_{w_0 x} : {}^{\mathrm{gr}}\widehat{\mathsf{L}}_{w_0 w}] \in \mathbb{Z}[t^{\pm 1}]$ is the (graded) composition multiplicity. The set of all ${}^p\underline{P}_w$ will be called the *periodic p-canonical basis* and the ${}^p p_{w,x}$ called the *periodic p-Kazhdan–Lusztig polynomials.*

The truth of this conjecture immediately gives the following (trivial) analogue to Theorem 4.

Corollary 4.5. *Assume $p > h$, then $[\mathsf{Z}_{w_0 x} : \widehat{\mathsf{L}}_{w_0 y}] = {}^p p_{x,y}(1)$ for any $x, y \in W_{\mathrm{aff}}$.*

Remark 4.6. In the case where $p \geq 2h-2$, another formula for these composition multiplicities has recently been obtained by S. Riche and G. Williamson in the forthcoming paper [RW1]. It is shown there that composition multiplicities are given by the values ${}^p h_{x,y}(1)$ for certain $x, y \in W_{\mathrm{aff}}$, where the ${}^p h_{x,y} \in \mathbb{Z}[t^{\pm 1}]$ denote the p-Kazhdan–Lusztig polynomials for the affine Hecke algebra (see [RW2]). In fact, it is likely that the arguments in [RW1] can be extended to the graded category. (Note that an algorithm for computing the ${}^p h_{x,y}$ is given in [JW].)

The construction of the periodic p-canonical basis, and the categorification of the periodic module above, are in terms of coherent sheaves on $\widetilde{\mathcal{N}}$. It should be possible to obtain similar results by considering a certain class of sheaves on the semi-infinite flag variety $\mathrm{Fl}^{\frac{\infty}{2}}_{G^\vee}$ (cf. [La, §3]). This space also admits a decomposition into Iwahori orbits, which are indexed by W_{aff} and are called the *semi-infinite Schubert cells.* As in the cases involving $\mathrm{Fl}_{G^\vee}$ and $\mathrm{Gr}_{G^\vee}$, one might expect to form a category

$$\mathcal{P}^{\frac{\infty}{2}} := \mathsf{Perv}^{\mathrm{mix}}_{(I^\vee)}(\mathrm{Fl}^{\frac{\infty}{2}}_{G^\vee})$$

of "mixed Iwahori constructible perverse sheaves" on $\mathrm{Fl}^{\frac{\infty}{2}}_{G^\vee}$. The baby Vermas (respectively co-baby Vermas) would then naturally correspond to the !-pushforwards (respectively $*$-pushforwards) of constant sheaves on the strata. Unfortunately, it is not possible to define this category in the obvious way since the Iwahori orbits have both infinite dimension *and* infinite codimension, and moreover, $\mathrm{Fl}^{\frac{\infty}{2}}_{G^\vee}$ does not have the structure of and ind-scheme.

So instead, $\mathcal{P}^{\frac{\infty}{2}}$ can be taken to be, informally, a category which naturally "simulates" the expected behavior of Iwahori-constructible perverse sheaves on $\mathrm{Fl}^{\frac{\infty}{2}}_{G^\vee}$. For instance, $\mathcal{P}^{\frac{\infty}{2}}$ could refer to the moment-graph model considered in [La]. The following conjecture can be regarded as an infinitesimal analogue to (1.1).

Conjecture 4.7. *There is an equivalence of categories $D^{\mathrm{b}}(\mathcal{P}^{\frac{\infty}{2}}) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{T\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$, where both categories "compatibly degrade" to $\mathsf{Rep}_\emptyset(\mathbf{G}_1\mathbf{T})$.*

4.2. **A diagrammatic approach.** In order to obtain a combinatorial algorithm for computing the composition multiplicities of the baby Vermas for $h < p < 2h-2$ as in [JW], it will first be necessary to obtain a diagrammatic presentation of $\mathsf{Rep}_\emptyset(\mathbf{G}_1\mathbf{T})$ (or $\widetilde{\mathsf{Rep}}_\emptyset(\mathbf{G}_1\mathbf{T})$). If $\mathbf{G} = GL_n(\Bbbk)$, then this should be possible by using similar methods to [RW2], where the tilting modules for $\mathbf{G}$ are replaced with injective modules for $\mathbf{G}_1\mathbf{T}$.

In fact, if combinatorial multiplicity formulas can be directly obtained for $p < 2h-2$ in this way, then it is not difficult to show that Donkin's lifting conjecture for tilting modules (see [Ja, E.9]), can be reduced to comparing the formulas obtained from this diagrammatic description to those of [RW1].

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